

ON LOCALLY SEMIPRIMITIVE GRAPHS AND A THEOREM OF WEISS

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ABSTRACT. In this paper we investigate graphs that admit a group acting arc-transitively such that the local action is semiprimitive with a regular normal nilpotent subgroup. This type of semiprimitive group is a generalisation of an affine group. We show that if the graph has valency coprime to six, then there is a bound on the order of the vertex stabilisers depending on the valency alone. We also prove a detailed structure theorem for the vertex stabilisers in the remaining case. This is a contribution to an ongoing project to investigate the validity of the Potočnik-Spiga-Verret Conjecture.

1. INTRODUCTION

All graphs in this paper are finite, connected and simple and every action of a group on a graph is faithful. If a group G acts on a graph Γ and x is a vertex of Γ , we write $\Gamma(x)$ for the neighbourhood of x in Γ and $G_x^{\Gamma(x)}$ for the permutation group induced on the set $\Gamma(x)$ by the stabiliser G_x . If \mathcal{P} is a property of permutation groups and L is some permutation group, we say that the pair (Γ, G) is *locally \mathcal{P}* , respectively, *locally L* to indicate that for all vertices x of Γ , $G_x^{\Gamma(x)}$ satisfies property \mathcal{P} , respectively, $G_x^{\Gamma(x)}$ is permutationally isomorphic to L .

Following [10], if there is a constant $c(L)$ such that for every locally L pair (Γ, G) we have $|G_x| \leq c(L)$, we say that L is *graph-restrictive*. In this language, the long-standing Weiss Conjecture [11] asserts that primitive permutation groups are graph-restrictive and the Praeger Conjecture asserts that quasiprimitive groups are graph-restrictive. Certain classes of permutation groups are known to be graph-restrictive: it is easy to see that regular groups are graph-restrictive for example. However the proof that 2-transitive groups are graph-restrictive [9, Theorem 1.4] is a deep result which uses the Classification of Finite Simple Groups. Fresh light was cast upon this problem by [5, Theorem 4] which shows that a graph-restrictive group is necessarily *semiprimitive* (see Definition 2.1). Going further, the authors of [5] stated the Potočnik-Spiga-Verret (PSV) Conjecture: a permutation group is graph-restrictive if and only if it is semiprimitive. Since the class of semiprimitive permutation groups properly encompasses the classes of primitive and quasiprimitive groups, the PSV Conjecture is broader in scope than both the Weiss and Praeger Conjectures and a proof of the former would imply the truth of the latter two conjectures.

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In this paper we investigate the validity of the PSV Conjecture for semiprimitive groups with a regular normal nilpotent subgroup. One can view this type of semiprimitive group as a generalisation of affine groups, which form one of the eight types of primitive groups. However our knowledge of semiprimitive groups pales in comparison to our knowledge of primitive groups, there is no O’Nan-Scott-Aschbacher type theorem for semiprimitive groups for example. In [1] soluble semiprimitive groups were studied and a classification achieved when the degree is square-free or a product of at most three primes. In Section 2 we strengthen some of the results of [1], related to semiprimitive groups with a regular normal nilpotent subgroup. Our first theorem is below. We remark that it is possible to obtain this theorem as a corollary to Theorem 1.2, but we offer a separate proof since a result proved along the way (Lemma 3.11) may be useful elsewhere and because it afforded us the opportunity to use such nice results as [4, 6.4.3] and [8].

Theorem 1.1. *Let Γ be a finite connected graph of valency d and let $G \leq \text{Aut}(\Gamma)$ be arc-transitive. Suppose that the local action is semiprimitive with a regular normal nilpotent subgroup and $(d, 6) = 1$. Then for each vertex x of Γ we have $|G_x| \leq d!(d-1)!$.*

In fact we arrive at the conclusion of the theorem above by showing that the group $G_{xy}^{[1]}$ is trivial, for some vertex y adjacent to x in Γ . Here, the group $G_{xy}^{[1]}$ is the kernel of the action of the edge stabiliser $G_{\{x,y\}}$ on the set $\Gamma(x) \cup \Gamma(y)$, it plays a prominent role in our investigations.

Our second theorem is a technical statement about the structure of vertex stabilisers. Weiss in [12] gave a detailed description of the structure of a vertex stabiliser in a locally affine graph. The most difficult part of Weiss’ proof is the application of so-called failure of factorisation arguments. With our weaker hypothesis it is difficult even to show that failure of factorisation results may be applied, since the local action may have more complicated structure from the outset. This is achieved in Lemma 3.14 which allows us to employ a result of Glauberman that delivers the following theorem which in the locally primitive case yields Weiss’ result. The notation will be explained in Section 3.

Theorem 1.2. *Let Γ be a finite connected graph and $G \leq \text{Aut}(\Gamma)$ be arc-transitive. Suppose that the local action is semiprimitive with a regular normal nilpotent subgroup and that $G_{xy}^{[1]} \neq 1$ for some edge $\{x, y\}$ of Γ . Let p be a prime dividing $|G_{xy}^{[1]}|$. Then there is a normal subgroup $L_x \leq G_x$ such that for $V = \langle \Omega Z(S) \mid S \in \text{Syl}_p(L_x) \rangle$ and $H := \mathbf{J}(L_x)C_{L_x}(V)/C_{L_x}(V)$ the following hold:*

- (a) $G_{xy}^{[1]}$ is a p -group and $p \in \{2, 3\}$.
- (b) $H = E_1 \times \cdots \times E_r$ and

$$V = C_V(H) \times [V, E_1] \times \cdots \times [V, E_r].$$

In particular, E_i acts faithfully on $[V, E_i]$ and trivially on $[V, E_j]$ for $j \neq i$.

- (c) $|[V, E_i]| = p^2$ and $E_i \cong \text{SL}_2(p)$ for $i = 1, \dots, r$.
- (d) $A = \times_{i=1}^r (A \cap E_i)$ and $|A|C_V(A) = |V|$ for all $A \in \mathcal{A}_V(H)$.

We also obtain the following corollary to Theorem 1.2 which highlights where possible counterexamples to the PSV Conjecture may be lurking.

Corollary 1.3. *Assume the hypothesis of Theorem 1.2 and let $K = G_x^{\Gamma(x)}$ with R the regular normal nilpotent subgroup of K . Then K contains normal subgroups J and F such that $F < R < J$ and one of the following holds:*

- (1) $J/F \cong \text{Sym}(3) \times \cdots \times \text{Sym}(3)$ when $p = 2$,
- (2) $J/F \cong \text{Alt}(4) \times \cdots \times \text{Alt}(4)$ when $p = 3$.

Below we provide some examples of semiprimitive groups which are either shown to be graph-restrictive by Theorem 1.1 or Corollary 1.3, or indicate areas where the PSV Conjecture is currently unknown.

Example 1.4. *Let q be an odd prime and let P be an abelian q -group. Let $H = C_2$ act on P by inversion and let $K = P \rtimes H$ act on K/H . Since H inverts every nontrivial element of P , Theorem 2.2 shows that K is semiprimitive in this action. For $q > 3$ Theorem 1.1 shows that K is graph-restrictive. For $q = 3$ the situation is quite different: since H normalises $\Phi(P)$ and inverts $P/\Phi(P)$ we see that every maximal subgroup M of P is a normal subgroup of K and that $K/M \cong \text{Sym}(3)$. Thus Corollary 1.3 does not provide any information in this case. If $\Phi(P) = 1$ it was shown in [3] that K is graph-restrictive, but for $\Phi(P) \neq 1$ this is still an open case of the PSV Conjecture.*

Example 1.5. *Let q be a prime, a, n and m integers and let $V = (\mathbb{F}_{q^a})^n$ be the n -dimensional vector space over \mathbb{F}_{q^a} . Let $H = \text{GL}(V)$ and let $W = V \oplus \cdots \oplus V$ be the direct sum of m copies of V . Set $K = W \rtimes H$ (where H acts in the natural way on each copy of V) and set $\Omega = W$, the set of vectors of W . Then K acts on Ω and W is a regular normal abelian subgroup. Theorem 2.2 shows that K is a semiprimitive group on Ω , and if $q > 3$ then this group is graph-restrictive by Corollary 1.3.*

If $q = 3$ then K has no normal section isomorphic to a direct product of groups isomorphic to $\text{Sym}(3)$ unless $n = 1$. Then K is just the extension of an elementary abelian 3-group by an involution which acts by inversion. As noted in the previous example, K is graph-restrictive in this case.

If $q = 2$ then for K to have a normal section isomorphic to a direct product of groups isomorphic to $\text{Alt}(4)$ we have to have $(n, a) = (2, 1)$ or $(n, a) = (1, 2)$. If $m = 1$ then K is isomorphic to $\text{Sym}(4)$ or $\text{Alt}(4)$ acting naturally on four points, so K is graph-restrictive by [2]. If $m > 1$ then this is an open case of the PSV Conjecture.

Example 1.6. *Let q be an odd prime, m an integer and let $E = q_+^{1+2m}$ be an extraspecial group. Since q is odd, $\text{Aut}(E)$ contains a subgroup $H \cong \text{Sp}_{2m}(q)$. Set $K = E \rtimes H$. Then H acts faithfully on E and both faithfully and irreducibly on $E/Z(E)$, so Theorem 2.2 shows that K is semiprimitive on the cosets of H . (These groups were shown to be semiprimitive in [1, Corollary 4.3].) For $q > 3$ we see that K is graph-restrictive by Theorem 1.1. For $q = 3$, since the only nontrivial normal subgroups of K contained in E are $Z(E)$ and E itself, we see there is no normal subgroup N of K such that either N or $N/Z(E)$ is isomorphic to a direct product of groups isomorphic to $\text{Sym}(3)$. Hence Corollary 1.3 shows that K is graph-restrictive in this case.*

Example 1.7. *Let $\pi = \{p_1, \dots, p_r\}$ be a finite set of primes such that $p_i \equiv -1 \pmod{3}$ for $i = 1, \dots, r$ and $p_1 < p_2 < \cdots < p_r$. For each $p_i \in \pi$ with $p_i > 2$ let V_i be an extraspecial*

group of plus type and order p_i^3 . If $p_1 = 2$ then let $V_1 = Q_8$. Let $H = \langle t \rangle \cong C_3$ act on each V_i as an element of order three in $\text{Aut}(V_i)$, note that H acts irreducibly on $V_i/Z(V_i)$.

Set $K := (V_1 \times \cdots \times V_r) \rtimes H$ and let K act on the cosets of H in K . Then K is semiprimitive by Theorem 2.2 with regular normal nilpotent subgroup $V_1 \times \cdots \times V_r$. If $p_1 > 2$ then K is graph-restrictive by Theorem 1.1. On the other hand, if $p_1 = 2$ then

$$K/(Z(V_1) \times V_2 \times \cdots \times V_r) \cong \text{Alt}(4),$$

and so Corollary 1.3 provides no information.

2. SOME RESULTS ON SEMIPRIMITIVE PERMUTATION GROUPS

Definition 2.1. Let G be a permutation group on Ω . A subgroup N of G is called *semiregular* if $N_\omega = 1$ for all $\omega \in \Omega$. We say that G is *semiprimitive* if G is transitive and every normal subgroup of G is transitive or semiregular.

In [1, Theorem 3.2] Bereczky and Maróti give necessary and sufficient conditions for a permutation group with a regular normal soluble subgroup to be semiprimitive. In the theorem below we remove the solubility part of the hypothesis. We recall some definitions before stating the result. Suppose that G is a group with subgroups H , K and L such that K and L are normal in G and $L \leq K$. Then H acts on the quotient K/L by $(kL)^h = k^hL$ for $h \in H$. The kernel of this action is the set $\{h \in H \mid [K, h] \leq L\}$ which is the largest normal subgroup M of H such that $[M, K] \leq L$.

Theorem 2.2. Let $G = K \rtimes H$. Then G is semiprimitive on the cosets of H if and only if H acts faithfully on every nontrivial H -invariant quotient of K .

Proof. Suppose that H acts faithfully on every H -invariant quotient of K and assume that $N \triangleleft G$. If $K \leq N$ then N acts transitively on G/H , so we may assume that $M := K \cap N$ is a proper subgroup of K . Since $N \triangleleft G$ we have that M is H -invariant, hence H acts on K/M . Since $[N, K] \leq N \cap K = M$ we see that N acts trivially on K/M , that is, $N \leq C_G(K/M)$. Hence

$$N \cap H \leq C_G(K/M) \cap H = C_H(K/M) = 1,$$

and so N is semiregular, as required.

Now assume that G is semiprimitive and let M be a proper normal subgroup of K which is H -invariant. Suppose that H does not act faithfully on K/M . Then $B := C_H(K/M)$ is a nontrivial normal subgroup of H . Moreover, K normalises BM since by definition $[K, B] \leq M$. Hence BM is a normal subgroup of G . Now $1 \neq B \leq BM \cap H$ so BM is not semiregular. If BM acts transitively on G/H then $G = BMH = MH$ and we have that $K = M(K \cap H) = M$, a contradiction. \square

Another way to phrase the above criterion is that for every nontrivial normal subgroup N of H we have $K = [K, N]$.

The following lemma is [1, Lemma 2.4] (see also [1, Corollary 3.1]), we offer a different proof which fits with our approach.

Lemma 2.3. *Suppose that G is a semiprimitive group with point stabiliser H and N is a normal intransitive subgroup of G . Then the action of G/N on HN/N is faithful and semiprimitive.*

Proof. We set $\overline{G} = G/N$ and use the bar notation. Let \overline{K} be a normal subgroup of \overline{G} . Then K is normal in G , so either K is transitive, which gives $KH = G$ and so $\overline{G} = \overline{KH}$ whence \overline{K} is transitive on $\overline{G}/\overline{H}$, or K is semiregular. The latter implies $K \cap H = 1$, which gives $K \cap HN = N(K \cap H) = N$, and therefore $\overline{K} \cap \overline{H} = 1$. Hence \overline{K} is semiregular, as required.

To see that \overline{G} is faithful on $\overline{G}/\overline{H}$, let $\overline{C} = \text{core}_{\overline{G}}(\overline{H})$. Then $N \leq C \leq HN$ and C is normal in G . If C were transitive, we would have $G = CH \leq HN$, which yields N transitive on G/H , a contradiction. Thus C is semiregular, so $C = C \cap HN = N(C \cap H) = N$, whence $\overline{C} = 1$ as required. \square

It was shown in [1, Theorem 3.1] that a soluble semiprimitive group has a unique regular normal subgroup and that this regular normal subgroup contains all semiregular subgroups and is contained in every transitive subgroup. Below we prove a result for semiprimitive groups which are not necessarily soluble, but we assume the existence of a regular normal soluble subgroup.

Theorem 2.4. *Let G be a semiprimitive group with a soluble regular normal subgroup K . Then every transitive normal subgroup of G contains K and every normal semiregular subgroup is contained in K . In particular, K is the unique regular normal subgroup.*

Proof. Let G and K be as in the statement and let H be a point stabiliser in G . We first prove that $C_G(K) \leq K$. Indeed, assume that $K \neq C_G(K)K$ and let $S \leq H$ be such that $KS = KC_G(K)$. Then S is normal in H and the solubility of K gives $[K, S] \leq [K, K] \neq K$. Now Theorem 2.2 shows that $S = 1$, a contradiction.

Now we assume that the theorem is false and let G be a counter-example of minimal order, so that G is a semiprimitive group which has a soluble regular normal subgroup K and T is another transitive normal subgroup. Set $M = T \cap K$, then since G is a counter-example to the theorem, $T \cap K \neq K$ so M is a proper subgroup of K . By the previous paragraph $[T, K] \neq 1$. Moreover $[T, K] \leq M < K$ so $1 \neq M$ is intransitive and semiregular. Now Lemma 2.3 shows that G/M acts semiprimitively on the cosets of HM/M where H is a point stabiliser in G . Since $THM = TH = G$, T/M is transitive on the cosets of H/M in G/M and $K \cap HM = M(K \cap H) = M$, so K/M is a regular normal soluble subgroup of G/M . Since $|G/M| < |G|$, G/M is not a counter-example to the theorem, so T/M contains K/M . This implies T contains K , a contradiction to G being a counter-example.

The second case is similar and is omitted. \square

The following example shows that the conclusion of the above theorem is false without the solubility hypothesis. Set

$$G = (T_1 \times T_2 \times T_3) : \langle x \rangle$$

where $T_1 \cong T_2 \cong T_3 \cong \text{Alt}(5)$ and x is an involution that acts as an outer automorphism on each T_i for $i = 1, 2, 3$. Let D be a full diagonal subgroup of $T_1 \times T_2 \times T_3$ which is

normalised by x and let $H = \langle D, x \rangle$. Let G act on G/H and note that T_1T_2 , T_2T_3 and T_1T_3 are three distinct regular normal subgroups. Since $G = T_1T_2 \rtimes H$ and H acts faithfully on T_1T_2 , T_1T_2/T_1 and on T_1T_2/T_2 , G is semiprimitive.

Lemma 2.5. *Let G be a semiprimitive group with a soluble regular normal subgroup K . Then every normal nilpotent subgroup is contained in K . In particular, $\mathbf{F}(G) = \mathbf{F}(K)$.*

Proof. Suppose that N is a normal nilpotent subgroup of G not contained in K . Since G is semiprimitive either N is semiregular or N is transitive. Theorem 2.4 shows that $K \leq N$ and N is transitive. Since N is nilpotent we have $[K, N] < K$, then $N \cap H \leq C_H(K/[K, N]) = 1$ by Theorem 2.2. Thus N is regular and $N = K$. \square

We note that our next lemma generalises [1, Theorem 3.4].

Lemma 2.6. *Let G be a semiprimitive group with point stabiliser H and regular normal nilpotent subgroup K . If there is a prime p such that $O_p(H) \neq 1$, then p does not divide $|K|$.*

Proof. Assume p divides $|K|$ and note that $N = O_{p'}(K)$ is a proper semiregular subgroup of K . By Lemma 2.3 the group G/N is semiprimitive with point stabiliser HN/N and regular normal subgroup K/N . Now K/N and $O_p(H)K/N$ are normal p -subgroups of G/N so Lemma 2.5 implies that $O_p(H)K/N = K/N$. This implies $O_p(H) \leq K \cap H = 1$, a contradiction. \square

3. LOCALLY SEMIPRIMITIVE GRAPHS WITH REGULAR NORMAL NILPOTENT SUBGROUPS

In this section we assume that G is a group acting faithfully and vertex-transitively on a connected finite graph Γ . Moreover we assume that the local action is semiprimitive with a regular normal nilpotent subgroup. We will prove Theorems 1.1 and 1.2.

If A is a set of vertices of Γ and $H \leq G_A$, we write H^A for the permutation group induced on A by H , in most cases A will be the neighbourhood of some vertex. We fix an edge $e = \{x, y\}$ of Γ and begin by assuming that $G_{xy}^{[1]} \neq 1$.

We will sometimes use the following two results without reference.

Proposition 3.1. *Let R be a subgroup of G_x and suppose that $R^{\Gamma(x)}$ is semiregular. Then $R \cap G_{xy} \leq G_x^{[1]}$.*

Proof. We have $R^{\Gamma(x)} \cap (G_{xy})^{\Gamma(x)} = 1$, that is $RG_x^{[1]} \cap G_{xy} \leq G_x^{[1]}$. The Dedekind identity gives

$$(R \cap G_{xy})G_x^{[1]} = RG_x^{[1]} \cap G_{xy} \leq G_x^{[1]}$$

and so $R \cap G_{xy} \leq G_x^{[1]}$ as required. \square

Lemma 3.2. *Suppose that $K \leq G_x \cap G_y$. If either (a) or (b) below hold, then $K = 1$.*

- (a) *The groups $N_{G_x}(K)^{\Gamma(x)}$ and $N_{G_y}(K)^{\Gamma(y)}$ are transitive.*
- (b) *The group $N_{G_x}(K)^{\Gamma(x)}$ is transitive and $N_{G_e}(K) \not\leq G_x \cap G_y$.*

Proof. Suppose that (a) holds and set $H = \langle N_{G_x}(K), N_{G_y}(K) \rangle \leq N_G(K)$. Since Γ is connected, H acts edge-transitively. Let u be any vertex of Γ and let v be adjacent to u . Then there exists $h \in H$ such that $\{x, y\}^h = \{u, v\}$. Now we obtain

$$K = K^h \leq (G_x \cap G_y)^h = G_u \cap G_v \leq G_u$$

whence K fixes every vertex of Γ , and therefore $K = 1$. The case that (b) holds is similar and is omitted. \square

Lemma 3.3. *There exists a prime p such that $G_{xy}^{[1]}$, $\mathbf{F}^*(G_x^{[1]})$ and $\mathbf{F}^*(G_{xy})$ are p -groups. In particular $G_{xy}^{[1]} \leq O_p(G_x^{[1]})$.*

Proof. This follows from [6, Corollary 3] and the fact that $\mathbf{F}^*(G_x^{[1]}) \leq \mathbf{F}^*(G_{xy})$. \square

We now establish some notation that will hold for the remainder of the paper. Recall that, for a p -group P , $\Omega Z(P)$ is the subgroup of $Z(P)$ generated by the elements of order p . We set

$$\begin{aligned} Q_x &= O_p(G_x^{[1]}), \\ L_x &= \langle (Q_x Q_y)^{G_x} \rangle, \\ R_0 &= O_{p'}(L_x), \\ Z_{xy} &= \Omega Z(Q_x Q_y), \\ Z_x &= \langle Z_{xy}^{G_x} \rangle. \end{aligned}$$

Lemma 3.4. *The following hold:*

- (i) $Q_y^{\Gamma(x)} \neq 1$;
- (ii) L_x is transitive on $\Gamma(x)$.

Proof. If (i) is false then $Q_y \leq G_x^{[1]}$ and so $Q_y = Q_x$. From Lemma 3.2 it follows that $Q_x = Q_y = 1$, a contradiction since $G_{xy}^{[1]} \leq Q_x \cap Q_y$ by Lemma 3.3. Clearly L_x is normal in G_x and by part (i) we have that $1 \neq Q_y^{\Gamma(x)} \leq L_x^{\Gamma(x)} \cap G_{xy}^{\Gamma(x)}$. Hence L_x is not semiregular, so $L_x^{\Gamma(x)}$ is transitive. \square

By Theorem 2.4, $L_x^{\Gamma(x)}$ contains the nilpotent regular normal subgroup of $G_x^{\Gamma(x)}$. We let R be the full pre-image of this subgroup, so $L_x \cap G_x^{[1]} \leq R \leq L_x$ and $R^{\Gamma(x)}$ is the nilpotent normal regular subgroup of $G_x^{\Gamma(x)}$.

Lemma 3.5. *The order of $R^{\Gamma(x)}$ is coprime to p .*

Proof. We have $Q_y^{\Gamma(x)}$ is a nontrivial normal subgroup of $G_{xy}^{\Gamma(x)}$ so the lemma follows from Lemma 2.6. \square

Lemma 3.6. *We have $O_p(G_x) = Q_x$, $C_{G_x}(Q_x)^{\Gamma(x)}$ is intransitive and*

$$C_{G_x}(Q_x) = Z(Q_x)O_{p'}(G_x).$$

Proof. We see that $O_p(G_x)^{\Gamma(x)}$ is a nilpotent normal subgroup of $G_x^{\Gamma(x)}$ so Lemma 3.5 and Lemma 2.5 show that $O_p(G_x)^{\Gamma(x)} = 1$. This gives $O_p(G_x) \leq G_x^{[1]}$ from which the first part of the lemma follows. For the third part, we just need to show that $C_{G_x}(Q_x) \leq Z(Q_x)O_{p'}(G_x)$ since the reverse inclusion is obvious.

Since $G_{xy}^{[1]}$ is nontrivial and is contained in $Q_x \cap Q_y$, it follows from Lemma 3.2 (b) that $C_{G_x}(Q_x)^{\Gamma(x)}$ is an intransitive normal subgroup of $G_x^{\Gamma(x)}$. Theorem 2.4 implies that $C_{G_x}(Q_x)^{\Gamma(x)} \leq R^{\Gamma(x)}$ and Lemma 3.5 shows that a Sylow p -subgroup P of $C_{G_x}(Q_x)$ is contained in $G_x^{[1]}$, whence $P \leq C_{G_x^{[1]}}(Q_x) = Z(Q_x)$ (by Lemma 3.3). Now we see that $|C_{G_x}(Q_x) : Z(Q_x)|$ is coprime to p , so the Schur-Zassenhaus Theorem gives $D \leq C_{G_x}(Q_x)$ such that $C_{G_x}(Q_x) = Z(Q_x)D \cong Z(Q_x) \times D$. Now $D = O_{p'}(C_{G_x}(Q_x)) \leq O_{p'}(G_x)$ and we are done. \square

Lemma 3.7. *We have $[L_x, G_x^{[1]}] \leq Q_x$, in particular, $L_x \cap G_x^{[1]}/Q_x \leq Z(L_x/Q_x)$.*

Proof. Since $G_x^{[1]} \leq G_y$ we see that $G_x^{[1]}$ normalises Q_y . Thus

$$[L_x, G_x^{[1]}] = [\langle (Q_x Q_y)^{G_x} \rangle, G_x^{[1]}] = \langle [Q_x Q_y, G_x^{[1]}]^{G_x} \rangle.$$

Now Q_x and Q_y normalise each other and $[G_x^{[1]}, Q_y] \leq G_x^{[1]} \cap Q_y \leq Q_x$, so

$$[Q_x Q_y, G_x^{[1]}] \leq Q_x.$$

This shows that $L_x \cap G_x^{[1]}/Q_x$ is contained in the centre of L_x/Q_x . \square

Lemma 3.8. *The group Q_x is the Sylow p -subgroup of R .*

Proof. By Lemma 3.7 we see that $L_x \cap G_x^{[1]}/Q_x$ is abelian. Since $O_p(L_x \cap G_x^{[1]}) \leq O_p(G_x^{[1]}) = Q_x$, we see that p does not divide $|L_x \cap G_x^{[1]} : Q_x|$. Since

$$|R : Q_x| = |R : L_x \cap G_x^{[1]}| |L_x \cap G_x^{[1]} : Q_x|$$

the result follows from Lemma 3.5. \square

Lemma 3.9. *We have $Z_x \leq \Omega Z(Q_x)$ and Q_x is the Sylow p -subgroup of $C_{L_x}(Z_x)$.*

Proof. Note that $[Z_{xy}, Q_x] \leq [Z_{xy}, Q_x Q_y] = 1$, so $Z_{xy} \leq C_{L_x}(Q_x)$. In particular, Z_{xy} is contained in a Sylow p -subgroup of $C_{G_x}(Q_x)$, which by Lemma 3.6 is equal to $Z(Q_x)$. Since Z_{xy} is elementary abelian, we have $Z_{xy} \leq \Omega Z(Q_x)$ and from this it follows that $Z_x \leq \Omega Z(Q_x)$.

For the second part we have that $Q_x \leq C_{L_x}(Z_x)$. If $C_{L_x}(Z_x) \leq L_x \cap G_x^{[1]}$ then we are done by Lemma 3.8. Otherwise, we see that $C_{L_x}(Z_x)$ is a normal subgroup of G_x which is not contained in $G_x^{[1]}$. Since $1 \neq Z_{xy}$ is centralised by $C_{L_x}(Z_x)$ we have that $C_{L_x}(Z_x)$ is semiregular on $\Gamma(x)$. Then $C_{L_x}(Z_x)(L_x \cap G_x^{[1]}) \leq R$ and so the result follows from Lemma 3.8. \square

Recall that a group X is p -separable if the series

$$1 \leq O_p(X) \leq O_{pp'}(X) \leq O_{pp'p}(X) \leq \cdots$$

terminates with X . Here the group $O_{pp'}(X)$ is defined by

$$O_{pp'}(X)/O_p(X) = O_{p'}(X/O_p(X))$$

and the other groups in the series are defined in the same recursive manner. A soluble group is p -separable for all primes p .

Lemma 3.10. *The following hold:*

- (i) $L_x = RQ_y$;
- (ii) $Q_xQ_y \in \text{Syl}_p(L_x)$;
- (iii) L_x is p -separable;
- (iv) if r is a prime dividing $|L_x \cap G_x^{[1]} : Q_x|$ then r divides $|R : R \cap G_x^{[1]}|$.

Proof. Since R is transitive on $\Gamma(x)$ we have that $\{Q_y^{G_x}\} = \{Q_y^R\}$. Whence

$$L_x = \langle (Q_xQ_y)^{G_x} \rangle = \langle (Q_xQ_y)^R \rangle \leq RQ_xQ_y = RQ_y$$

and since $RQ_y \leq L_x$ we have equality so (i) holds. By Lemma 3.8 we have $R \cap Q_y = Q_x \cap Q_y$ and it follows that $Q_xQ_y \in \text{Syl}_p(L_x)$ which is (ii). We observe that $Q_x = O_p(L_x)$, $R = O_{pp'}(L_x)$ and $L_x = O_{pp'p}(L_x)$ which gives (iii).

Finally suppose that r is a prime dividing $|L_x \cap G_x^{[1]} : Q_x|$ and let E be a Sylow r -subgroup of $L_x \cap G_x^{[1]}$. Let $\overline{L_x} = L_x/Q_x$, then \overline{E} is a nontrivial central subgroup of $\overline{L_x}$ by Lemma 3.7. If r does not divide $|R : L_x \cap G_x^{[1]}|$, then by part (1) \overline{E} is a Sylow r -subgroup of $\overline{L_x}$, and so there is a normal complement \overline{F} to \overline{E} in $\overline{L_x}$ by Burnside's Normal p -Complement Theorem. We have $\overline{Q_xQ_y} \leq \overline{F}$ and therefore $\overline{L_x} = \langle \overline{Q_xQ_y}^{\overline{L_x}} \rangle \leq \overline{F}$. Now $\overline{E} \leq \overline{E} \cap \overline{F} = 1$, a contradiction. \square

The next lemma implies that $O_p(L_x/O_{p'}(L_x)) = O_p(L_x)O_{p'}(L_x)/O_{p'}(L_x)$. This is not standard behaviour for p -separable groups, indeed, the group $X := \text{Sym}(3) \times C_2$ is 2-separable, but

$$O_2(X/O_3(X)) = X/O_3(X) \neq O_2(X)O_3(X)/O_3(X).$$

Lemma 3.11. *Let $\overline{L_x} = L_x/R_0$. Then $O_p(\overline{L_x}) = \overline{Q_x}$.*

Proof. Let $U \leq L_x$ be such that $\overline{U} = O_p(\overline{L_x})$ and note that $\overline{Q_x} \leq \overline{U}$ so $Q_x \leq U$. Choose U_0 to be a Sylow p -subgroup of U so that $U = R_0U_0$ and we may assume that $U_0 \leq Q_xQ_y$ by Lemma 3.10. Note that $Q_x \leq U_0$ and (since $R_0 \leq R$) we have

$$U \cap R = R_0U_0 \cap R = R_0(U_0 \cap R) = R_0Q_x,$$

where the last equality is by Lemma 3.8. Now $[U_0, R] \leq [U, R] \leq U \cap R = R_0Q_x < R$ where the last inequality holds since Lemma 3.6 shows that R_0Q_x is intransitive but R is transitive by definition. By our choice of R , $R^{\Gamma(x)}$ is the normal nilpotent regular subgroup of $(G_x)^{\Gamma(x)}$. Then

$$[(U_0)^{\Gamma(x)}, R^{\Gamma(x)}] = [U_0, R]^{\Gamma(x)} \leq (Q_xR_0)^{\Gamma(x)} = (R_0)^{\Gamma(x)} < R^{\Gamma(x)}.$$

The calculation above shows that $(U_0)^{\Gamma(x)}$ centralises the nontrivial quotient $R^{\Gamma(x)}/(R_0)^{\Gamma(x)}$. On the other hand, since R_0 is normal in G_x , $(R_0)^{\Gamma(x)}$ is a $(G_{xy})^{\Gamma(x)}$ -invariant normal

subgroup of $R^{\Gamma(x)}$ and Theorem 2.2 says that $(G_{xy})^{\Gamma(x)}$ acts faithfully on $R^{\Gamma(x)}/(R_0)^{\Gamma(x)}$. Hence $(U_0)^{\Gamma(x)} = 1$, that is, $U_0 \leq L_x \cap G_x^{[1]}$ and therefore $U_0 \leq Q_x$. Hence $\overline{U} \leq \overline{Q_x}$ as required. \square

Lemma 3.12. *We have $p \in \{2, 3\}$ and $q := 5 - p$ divides $|R/L_x \cap G_x^{[1]}|$.*

Proof. Let $\overline{L_x} = L_x/R_0$ and note that by [4, 6.4.3] $\overline{L_x}$ has characteristic p . If $p \geq 5$ then $\overline{L_x}$ is p -stable by [4, 9.4.5 (1)]. If $p = 3$ and $q \nmid |R/L_x \cap G_x^{[1]}|$ then $\overline{L_x}$ has odd order by Lemma 3.10 (iv), and is therefore p -stable by [4, 9.4.5 (2)]. If $p = 2$ and $q \nmid |R/L_x \cap G_x^{[1]}|$ then $\overline{L_x}$ has order coprime to three by Lemma 3.10 and is therefore $\text{Sym}(4)$ -free.

Suppose for a contradiction that one of the first two cases holds. Then we may apply [4, 9.4.4 (b)] to $\overline{L_x}$ to obtain a nontrivial characteristic subgroup \overline{D} of $\overline{Q_x Q_y}$ which is normal in $\overline{L_x}$. Lemma 3.11 gives $\overline{D} \leq \overline{Q_x}$. Since the preimage of $\overline{Q_x}$ is $Q_x R_0 \cong Q_x \times R_0$ we may choose a subgroup D of Q_x such that D has image \overline{D} . Since \overline{D} is normal in $\overline{L_x}$ we see that DR_0 is normal in L_x and $D \in \text{Syl}_p(DR_0)$. The Frattini argument shows that $L_x = DR_0 N_{L_x}(D)$. Since $[R_0, D] \leq [R_0, Q_x] = 1$ we have that D is normal in L_x . Now $\overline{Q_x Q_y}$ is isomorphic to $Q_x Q_y$, so D is characteristic in $Q_x Q_y$. But now $1 \neq D$ is normalised by $\langle L_x, G_{\{x,y\}} \rangle$, a contradiction.

Suppose now that the third case holds. Since $\overline{L_x}$ is now $\text{Sym}(4)$ -free, we may use [8] in place of [4, 9.4.4 (b)] and a similar argument to above leads to a contradiction. \square

Proof of Theorem 1.1. Let Γ and G be as in the hypothesis of Theorem 1.1. Since the index of $G_{xy}^{[1]}$ in G_x is at most $d!(d-1)!$ we assume for a contradiction that $G_{xy}^{[1]} \neq 1$. In particular, we may apply all of the results in this section. Let $N^{\Gamma(x)}$ be the regular normal nilpotent subgroup of $(G_x)^{\Gamma(x)}$, and note that $d = |N^{\Gamma(x)}|$ is coprime to six. Using the notation of Lemma 3.12 we have

$$N^{\Gamma(x)} \cong R/(L_x \cap G_x^{[1]}).$$

Lemma 3.12 shows that either $2 \mid |N^{\Gamma(x)}|$ or $3 \mid |N^{\Gamma(x)}|$, a contradiction. \square

In the next proposition we use the Thompson subgroup. For a p -group S we let $\mathcal{A}(S)$ be the set of elementary abelian subgroups of maximal order in S . Then the *Thompson subgroup* of S is

$$\mathbf{J}(S) = \langle A \mid A \in \mathcal{A}(S) \rangle.$$

For a group F and a prime p we set

$$\mathbf{J}(F) = \langle \mathbf{J}(S) \mid S \in \text{Syl}_p(F) \rangle.$$

Proposition 3.13. *With $\overline{L_x} = L_x/R_0$ we have $\mathbf{J}(\overline{L_x}) = \overline{\mathbf{J}(L_x)}$.*

Proof. Let S be a Sylow p -subgroup of L_x . Since R_0 has order coprime to p we see that $\overline{S} \in \text{Syl}_p(\overline{L_x})$ and $\mathbf{J}(\overline{S}) = \overline{\mathbf{J}(S)}$. Hence

$$\mathbf{J}(\overline{L_x}) = \langle \mathbf{J}(\overline{S}) \mid \overline{S} \in \text{Syl}_p(\overline{L_x}) \rangle = \langle \overline{\mathbf{J}(S)} \mid S \in \text{Syl}_p(L_x) \rangle = \overline{\mathbf{J}(L_x)}.$$

\square

Following [4, 9.2] we say that a group F is Thompson factorizable with respect to the prime p if p divides $|F|$ and for some Sylow p -subgroup S of F we have

$$F = O_{p'}(F)C_F(\Omega Z(S))N_F(\mathbf{J}(S)).$$

We use this notion in the next lemma.

Lemma 3.14. *Let $\overline{L_x} = L_x/R_0$. Then $O_{p'}(\overline{L_x}) = 1$ and $\overline{L_x}$ is not Thompson factorizable with respect to p .*

Proof. Since $R_0 = O_{p'}(L_x)$ the first part of the claim holds trivially. Suppose the latter part of the claim is false and set

$$\overline{V} = \langle \Omega Z(\overline{S}) \mid \overline{S} \in \text{Syl}_p(\overline{L_x}) \rangle.$$

By our assumption that $\overline{L_x}$ is Thompson factorizable, [4, 9.2.12] implies

$$\mathbf{J}(\overline{L_x}) \leq C_{\overline{L_x}}(\overline{V}).$$

Using Lemmas 3.4 and 3.10 (ii) and the fact that $|R_0|$ is coprime to p we have

$$\overline{V} = \langle \Omega Z(\overline{Q_x Q_y})^{\overline{L_x}} \rangle = \overline{\langle \Omega Z(Q_x Q_y)^{L_x} \rangle} = \overline{Z_x}.$$

By Proposition 3.13 we have $\mathbf{J}(\overline{L_x}) = \overline{\mathbf{J}(L_x)}$ and so $\mathbf{J}(\overline{Q_x Q_y}) = \overline{\mathbf{J}(Q_x Q_y)} \leq C_{\overline{L_x}}(\overline{Z_x})$. Again using that $|R_0|$ is coprime to p , we have that $C_{\overline{L_x}}(\overline{Z_x}) = \overline{C_{L_x}(Z_x)}$. Hence $\overline{\mathbf{J}(Q_x Q_y)} \leq \overline{C_{L_x}(Z_x)}$ implies that

$$R_0 \mathbf{J}(Q_x Q_y) \leq R_0 C_{L_x}(Z_x) = C_{L_x}(Z_x).$$

By Lemma 3.9, Q_x is a Sylow p -subgroup of $C_{L_x}(Z_x)$, whence $\mathbf{J}(Q_x Q_y) \leq Q_x$. We obtain $\mathbf{J}(Q_x Q_y) = \mathbf{J}(Q_x)$, a contradiction. \square

Let $J_x = \mathbf{J}(L_x)C_{L_x}(Z_x)$. Note that $Q_x R_0 \leq C_{L_x}(Z_x) \leq J_x$.

Proposition 3.15. *The group $J_x^{\Gamma(x)}$ is transitive.*

Proof. Clearly J_x is normal in G_x , so if the result is false then J_x is semiregular on $\Gamma(x)$. Then $\mathbf{J}(Q_x Q_y) \leq J_x \cap G_{xy} \leq G_x^{[1]}$ and from this it follows that $\mathbf{J}(Q_x Q_y) = \mathbf{J}(Q_x)$, a contradiction. \square

We can now prove Theorem 1.2 which we restate for convenience and make the substitution $Z_x = V$.

Theorem 1.2. *Let $H = J_x/C_{L_x}(Z_x)$. Then the following hold:*

- (a) $p \in \{2, 3\}$.
- (b) $H = E_1 \times \cdots \times E_r$ and

$$Z_x = C_{Z_x}(H) \times [Z_x, E_1] \times \cdots \times [Z_x, E_r].$$

In particular, E_i acts faithfully on $[Z_x, E_i]$ and trivially on $[Z_x, E_j]$ for $j \neq i$.

- (c) $||[Z_x, E_i]| = p^2$ and $E_i \cong \text{SL}_2(p)$ for $i = 1, \dots, r$.
- (d) $A = \times_{i=1}^r (A \cap E_i)$ and $|A| |C_{Z_x}(A)| = |Z_x|$ for all $A \in \mathcal{A}_{Z_x}(H)$.

Proof. By Lemma 3.14 we may apply [4, 9.3.8] to $\overline{L_x} = L_x/R_0$ which yields statements (a)-(d) for $\mathbf{J}(\overline{L_x})\mathbf{C}_{\overline{L_x}}(\overline{Z_x})/\mathbf{C}_{\overline{L_x}}(\overline{Z_x})$. By Proposition 3.13 we have $\mathbf{J}(\overline{L_x}) = \overline{\mathbf{J}(L_x)}$ and since $|R_0|$ is coprime to p we have $\mathbf{C}_{\overline{L_x}}(\overline{Z_x}) = \overline{\mathbf{C}_{L_x}(Z_x)}$. Furthermore, since $[R_0, Z_x] \leq [R_0, Q_x] = 1$ we see $R_0 \leq \mathbf{C}_{L_x}(Z_x)$. Hence

$$\mathbf{J}(\overline{L_x})\mathbf{C}_{\overline{L_x}}(\overline{Z_x})/\mathbf{C}_{\overline{L_x}}(\overline{Z_x}) = \overline{\mathbf{J}(L_x)\mathbf{C}_{L_x}(Z_x)/\mathbf{C}_{L_x}(Z_x)} \cong \mathbf{J}(L_x)\mathbf{C}_{L_x}(Z_x)/\mathbf{C}_{L_x}(Z_x)$$

and so the statement above holds. \square

Proof of Corollary 1.3. Theorem 1.2 gives information about $J_x/\mathbf{C}_{L_x}(Z_x)$. We now convert this into information about $J_x/(J_x \cap G_x^{[1]}) \cong J_x^{\Gamma(x)}$. Write $J_x^{[1]} := J_x \cap G_x^{[1]}$, $M_x := \mathbf{C}_{L_x}(Z_x)$ and note that $J_x/J_x^{[1]}$ contains the normal subgroup $M_x J_x^{[1]}/J_x^{[1]}$. Since $J_x^{\Gamma(x)}$ is transitive by Proposition 3.15, we may choose a subgroup R of J_x so that $J_x^{[1]} \leq R \leq J_x$ and $R^{\Gamma(x)}$ is the nilpotent regular normal subgroup of G_x . Since $M_x^{\Gamma(x)}$ is intransitive, we have $M_x \leq M_x J_x^{[1]} \leq R$. The quotient $J_x/M_x J_x^{[1]}$ is visible as a quotient of

$$J_x/M_x \cong E_1 \times \cdots \times E_r$$

where each E_i is isomorphic to $\mathrm{SL}_2(p)$ with $p = 2$ or $p = 3$. Lemma 3.7 yields $[J_x, J_x^{[1]}] \leq [L_x, L_x \cap G_x^{[1]}] \leq Q_x$, so we obtain

$$[J_x, M_x J_x^{[1]}] \leq [J_x, M_x][J_x, J_x^{[1]}] \leq M_x Q_x \leq M_x,$$

whence $M_x J_x^{[1]}/M_x \leq \mathbf{Z}(J_x/M_x)$. Choose $F \leq J_x$ with $M \leq F$ so that $F/M_x = \mathbf{Z}(J_x/M_x)$. Then

$$J_x^{[1]} \leq M_x J_x^{[1]} \leq F \leq J_x,$$

F is normal in G_x and J_x/F is isomorphic to a direct product of groups isomorphic to $\mathrm{PSL}_2(p)$ where $p = 2$ or $p = 3$. This gives the corollary. \square

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